GAUGE FIELD THEORY FOR POINCARÉ–WEYL GROUP

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Abstract

On the basis of the general principles of a gauge field theory the gauge theory for the Poincaré–Weyl group is constructed. It is shown that tetrads are not true gauge fields, but represent functions from true gauge fields: Lorentzian, translational and dilatational ones. The equations of gauge fields which sources are an energy-momentum tensor, orbital and spin momemta, and also a dilatational current of an external field are obtained. A new direct interaction of the Lorentzian gauge field with the orbital momentum of an external field appears, which describes some new effects. Geometrical interpretation of the theory is developed and it is shown that as a result of localization of the Poincaré–Weyl group spacetime becomes a Weyl–Cartan space. Also the geometrical interpretation of a dilaton field as a component of the metric tensor of a tangent space in Weyl–Cartan geometry is proposed.

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1. Introduction

It is well-known that the gauge treatment of physical interactions underlies modern fundamental physics [1], [2]. Application of the gauge approach to gravitational interactions was developed in [3]–[4] for the Lorentz group and in [5]–[9] for the Poincaré group (see also reviews [10]–[14], the book [15] and the literature cited there). Nevertheless, up to present time the interest to the gauge treatment of gravitational interaction [13]–[18] does not weaken. However, the first gauge theory was offered yet in 1918 by Weyl (see [19]), who introduced a gauge field appropriate to the group of changes of scales (calibres), which were arbitrarily in each point of spacetime. Change of length scales is equivalent in mathematical sense to expansion or compression (dilatations) of space. Connection of the dilatation group with the Poincaré group results in expansion of the Poincaré group to the Poincaré–Weyl group.

Importance of consideration of the Poiancáre–Weyl group is connected with that role, which the Weyl's scale symmetry plays in a quantum field theory. Violation of this symmetry at a quantum level results in occurrence of the Weyl's anomaly connected with the problems of definition of contr-terms structure and asymptotic freedom in the quantum field theory, with supersymmetry, with calculation of critical dimensions $n = 26$ and $n = 10$ in a strings theory, with gravitational instantons, with the Hoking's phenomenon of evaporation of black holes, with the problems of inflation, the cosmological constant, birthes of particles and black holes in the early universe [20]. For research of some of the listed problems, application of known techniques of BRST-symmetry [21], [22] to Weyl's gauge scale transformations is used [23].

Construction of a gauge theory for the Poincaré–Weyl group was developed in [24],[25]. According to our opinion, lack of these works consists in the idea (ascending to the work of Kibble [5]) that the gauge fields for the group of translations are tetrads $h^a{}_\mu$. This point of view obviously contradicts the fact that gauge fields should not be transformed as tensors at gauge transformations while the tetrads are transformed as tensor components on both indexes: tetrad and coordinate ones. We note that in [11], [13] it was pointed out on illegitimacy of the treatment of tetrads as gauge fields.

In the present work the construction of the gauge theory for the Poincaré–Weyl group[1] is made, deprived the specified lack. In the basis of this construction there is the method

[1]The given extension of the Poincaré group sometimes is named the Weyl group [24]. From our point of view the name 'Poincaré–Weyl group' is more exact, as it is the locale scale transformation that is usually connected with the concept of the Weyl's symmetry.

of introduction of gauge fields for the groups connected to transformations of spacetime coordinates, developed in [6], [7], [15]. The first and the second Noether theorems are used that allows to introduce the gauge fields, dynamically realizing the appropriate conservation laws. In the offered approach the quantities $h^a_{\ \mu}$ are not gauge fields, but represent some functions from the true gauge fields. The understanding of what gauge fields are the true potentials of a gravitational field is obviously important in view of the realization of quantization procedure for the gravitational field understood as a gauge field for the Poincaré–Weyl group. Besides, within the framework of general gauge procedure, the scalar Dirac's field [26] and the Utiyama's 'measure' scalar field [27] are naturally introduced. These scalar fields play an essential role for construction of a gravitational field Lagrangian.

The paper is organized as follows. In section 2 the Poincaré–Weyl group and its action on physical fields are discussed. In section 3 the Noether theorem for the Poincaré–Weyl group is formulated and there are appropriate laws of conservation of an energy-momentum, spin and orbital momemta, and also a dilatational current. In section 4, following [15], four initial postulates of the theory are formulated: the principle of local invariance, the principle of stationary action, the principle of minimality of gauge interactions, and the postulate on existence of a free gauge field. In section 5 the gauge invariant Lagrangian of the interaction of external and gauge fields is derived from main principles. A new direct interaction of the gravitational gauge field with the orbital momentum of an external field appears, which describes some new effects. In section 6 the gauge invariant Lagrangian of free gauge fields and the gauge fields equations are derived. At last, in section 7 a geometrical interpretation of the theory is made and it is found out that at localization of the Poincaré–Weyl group there is a space with a Weyl–Cartan geometry. In the Conclusion the basic results of the work are discussed, and the resulted Lagrangian of the gravitational field is proposed, all gauge fields of the gravitational field gauge theory considered allowing to be dynamically realized.

2. The Poincaré–Weyl group

Let spacetime $\mathcal{M}$ with a metric tensor $\breve{g}$ has structure of a flat space which geometry is defined according to F. Klein's Erlangen program by the global definition of action of the Poincaré–Weyl group $\mathcal{PW}(\omega\ \varepsilon\ a)$ as a fundamental group (on E. Cartan's terminology [28]) of this geometry. The fundamental group determines geometry of space as system invariants, that is relations and geometrical images, which remain constant in space at

action of the given group (see [29]). The similar geometry arises in spacetime filled with radiation and ultra relativistic particles. The space is flat, when curvature, torsion and nonmetricity tensors are equal to zero at all space. Such type of space $\mathcal{M}$ is appropriately to name as Minkowski–Weyl space.

On $\mathcal{M}$ a special system of coordinates x^i $(i = 1, 2, 3, 4)$ (analogue of the Cartesian coordinates in Minkowski space) can be globally introduced, in which metric tensor components are equal

$$g_{ij} = \beta^2 g^M_{ij} \ , \quad \beta = const > 0 \ , \quad g^M_{ij} = \mathrm{diag}\,(1,\ 1,\ 1,\ -1) \ , \tag{2.1}$$

where g^M_{ij} are components of the metric tensor of Minkowski space.

We represent infinitesimal transformations of the group $\mathcal{PW}$ as follows:

$$\delta x^i = \omega^m I_m{}^i{}_j\, x^j - \varepsilon x^i + a^i = -(\omega^m \overset{\circ}{M}_m + \varepsilon \overset{\circ}{D} + a^k P_k) x^i \ , \tag{2.2}$$

$$\overset{\circ}{M}_m = -I_m{}^l{}_j\, x^j \frac{\partial}{\partial x^l} \ , \quad I_m{}^{ij} = I_m{}^{[ij]} \ , \quad \overset{\circ}{D} = x^l \frac{\partial}{\partial x^l} \ , \quad P_k = -\,\delta^l_k \frac{\partial}{\partial x^l} \ . \tag{2.3}$$

Introducing the generalized designation $\{\omega^z\} = \{\omega^m\,,\,\varepsilon\,,\,a^k\}$ for a set of parameters of transformations of the given group, the transformation (2.2) is convenient to represent as

$$\delta x^i = \omega^z X^i_z \ , \quad X^i_m = I_m{}^i{}_j\, x^j \ , \quad X^i = -\,x^i \ , \quad X^i_k = \delta^i_k \ . \tag{2.4}$$

where $I_\emptyset{}^i{}_j = -\,\delta^i_j$, and $\emptyset$ is a symbol of empty set.

Operators $\overset{\circ}{M}_m$ and P_k are generators of 4-rotations (Lorentz subgroup L_4) and 4-shifts (a subgroup of translations T_4), and the operator $\overset{\circ}{D}$ is a generator of dilatation (a subgroup of dilatation D_4) of the space $\mathcal{M}$. The given operators satisfy to the following commutation relations:

$$\begin{gathered}
\left[\overset{\circ}{M}_m\,,\,\overset{\circ}{M}_n\right] = \ c_m{}^q{}_n \ \overset{\circ}{M}_q \ , \qquad \left[\overset{\circ}{M}_m\,,\,P_k\right] = I_m{}^l{}_k P_l \ , \qquad [P_k\,,\,P_l] = 0 \ , \\
\left[\overset{\circ}{M}_m\,,\,\overset{\circ}{D}\right] = 0 \ , \qquad \left[P_k\,,\,\overset{\circ}{D}\right] = P_k \ , \qquad \left[\overset{\circ}{D}\,,\,\overset{\circ}{D}\right] = 0 \ .
\end{gathered} \tag{2.5}$$

Let an arbitrary field ψ^A is given on $\mathcal{M}$, the infinitesimal transformation of which under action of the group $\mathcal{PW}(\omega,\,\varepsilon,\,a)$ looks like

$$\delta\psi^A = \omega^m I_m{}^A{}_B \psi^B + \varepsilon w \psi^A = \omega^z I_z{}^A{}_B \psi^B \ , \quad I_z{}^A{}_B = \{I_m{}^A{}_B\,,\,I_\emptyset{}^A{}_B\} \ , \quad I_\emptyset{}^A{}_B = w\delta^A_B \ , \tag{2.6}$$

where w is a weight of ψ^A under the action of the subgroup of dilatation D_4. Operators $I_m{}^A{}_B$ satisfy to commutation relations: $I_m{}^A{}_C\, I_n{}^C{}_B - I_n{}^A{}_C\, I_m{}^C{}_B = c_m{}^q{}_n\, I_q{}^A{}_B$.

Action of group $\mathcal{PW}$ on metric tensor occurs as follows:

$$\delta g_{ij} = -\,\omega^z\, I_z{}^l{}_i\, g_{lj} - \omega^z\, I_z{}^l{}_j\, g_{il} = -\,2\omega^z\, I_{z(ij)} = 2\varepsilon\, g_{ij} \ . \tag{2.7}$$

Components g_{ij} are not invariant under the action of $\mathcal{PW}$. As a result of (2.1), under the action of $\mathcal{PW}$ the following transformation holds, $\beta^2 \longrightarrow \beta^2 + 2\varepsilon$, $\delta\beta = \varepsilon\beta = \omega^z \delta_z^{\varnothing} \beta$.

We introduce on $\mathcal{M}$ an arbitrary curvilinear system of coordinates $\{x^\mu\} = \{x^\mu(x^i)\}$:

$$dx^i = \overset{\circ}{h}{}^i{}_\mu dx^\mu \ , \qquad \overset{\circ}{h}{}^i{}_\mu = \partial_\mu x^i \ , \qquad \partial_\mu = \frac{\partial}{\partial x^\mu} \ . \tag{2.8}$$

Then for metric tensor it is valid:

$$\begin{aligned} &ds^2 = \overset{\circ}{g}_{\mu\nu} \, dx^\mu dx^\nu \ , \qquad \overset{\circ}{g}_{\mu\nu} = \breve{g}(\vec{e}_\mu \, , \, \vec{e}_\nu) = g_{ij} \, \overset{\circ}{h}{}^i{}_\mu \, \overset{\circ}{h}{}^j{}_\nu \ , \\ &\overset{\circ}{g} = \det(\overset{\circ}{g}_{\mu\nu}) = \det(g_{ij}) \, (\overset{\circ}{h})^2 \ , \quad \overset{\circ}{h} = \det(\overset{\circ}{h}{}^a{}_\mu) \ . \end{aligned} \tag{2.9}$$

The curvilinear system of coordinates in flat space is introduced with the purpose to separate the problems connected with invariance of the theory under gauge transformations, and the problems following from the requirement of covariance of the theory concerning the group of the general transformations of coordinates. In works [5]–[7], [24], [25], the curvilinear system of coordinates was not introduced before procedure of localization. As a result, transformations of coordinates under action of gauge groups after localization became the general transformations of coordinates that broke mathematical structure of gauge groups.

As a result of (2.6) and (2.7), both $\psi^A(x)$ and g_{ij} are not transformed as representation of the subgroup of translations T_4, but change under action of T_4 only owing to transformation of argument x^k. Therefore an action of the operator of shift on $\mathcal{M}$, for example, on a field $\psi^A(x)$ (when $\delta x^k = a^k$) is realized as follows:

$$\psi^A(x + \delta x) = \psi^A(x) + \delta x^\mu \frac{\partial}{\partial x^\mu} \psi(x) = \psi^A(x) - a^k P_k \psi^A(x) \ , \tag{2.10}$$

$$P_k = - \, \overset{\circ}{h}{}^\mu{}_k \, \partial_\mu \ , \qquad \overset{\circ}{h}{}^\mu{}_k \, \overset{\circ}{h}{}^k{}_\nu = \delta^\mu_\nu \ . \tag{2.11}$$

Under the action of $\mathcal{PW}(\omega \, , \, \varepsilon \, , \, a)$, variation $\bar{\delta}$ of a form of the field function ψ^A:

$$\begin{aligned} \bar{\delta}\psi^A &= \psi'^A(x) - \psi^A(x) = \delta\psi^A - \delta x^\mu \partial_\mu \psi^A = \delta\psi^A + \delta x^k P_k \psi^A = \\ &= \delta\psi^A + \omega^z X_z^k P_k \psi^A = \delta\psi^A - \omega^z X_z^k \, \overset{\circ}{h}{}^\mu{}_k \partial_\mu \psi^A \ , \end{aligned} \tag{2.12}$$

commutes with the operator of differentiation. The following commutation relations hold:

$$[\bar{\delta} \, , \, P_k] = 0 \ , \qquad [\delta \, , \, P_k] = (P_k \delta x^l) \, P_l \ , \tag{2.13}$$

which are valid owing to identities: $\delta \, \overset{\circ}{h}{}^\mu{}_k = - \, \overset{\circ}{h}{}^\mu{}_l \, \overset{\circ}{h}{}^\nu{}_k \, \delta \, \overset{\circ}{h}{}^l{}_\nu$ and

$$\bar{\delta} \, \overset{\circ}{h}{}^l{}_\nu = \partial_\nu \delta x^l - \overset{\circ}{h}{}^l{}_\mu \partial_\nu \delta x^\mu - \delta x^\mu \partial_\mu \overset{\circ}{h}{}^l{}_\nu = 2 \partial_{[\nu} \overset{\circ}{h}{}^l{}_{\mu]} \, \delta x^\mu = 0 \ .$$

3. Noether theorem for the Poincaré–Weyl group

The field ψ^A on $\mathcal{M}$ is described in curvilinear coordinates by the action

$$J = \int_\Omega (dx)\,\mathcal{L}\,, \qquad \mathcal{L} = \sqrt{|\overset{\circ}{g}|}\,L(\psi^A,\, P_k\psi^A,\, \beta^2 g^M_{ij})\,. \tag{3.1}$$

Under transformations of $\mathcal{M}$ under the action of the Poincaré–Weyl group (2.2), variation of action integral with regard of change of integration area Ω is equal

$$\delta J = \delta\int_\Omega (dx)\,\mathcal{L} = \int_\Omega (dx)\Big(\sqrt{|\overset{\circ}{g}|}\,(\partial_\mu\delta x^\mu)\,L + \delta\sqrt{|\overset{\circ}{g}|}\,L + \sqrt{|\overset{\circ}{g}|}\,\delta L\Big) = 0\,. \tag{3.2}$$

In curvilinear system of coordinates, as consequence of (2.9) an equality holds, $\delta\sqrt{|\overset{\circ}{g}|} = -\sqrt{|\overset{\circ}{g}|}\,(\partial_\mu\delta x^\mu)$. Therefore (3.2) by virtue of randomness of area Ω means $\delta L = 0$:

$$\delta L = \frac{\partial L}{\partial\psi^A}\delta\,\psi^A + \frac{\partial L}{\partial P_k\psi^A}\,\delta P_k\psi^A + \frac{\partial L}{\partial\beta}\,\delta\beta = 0\,.$$

Calculating the variations on the basis of (2.6)–(2.13) and using equality $P_k X^i_z = -I_z{}^i{}_k$, we receive as consequence of randomness of ω^z the following identity:

$$\frac{\partial L}{\partial\psi^A} I_z{}^A{}_B\psi^B + \frac{\partial L}{\partial P_k\psi^A}(I_z{}^A{}_B P_k\psi^B - I_z{}^l{}_k P_l\psi^A) + \frac{\partial L}{\partial\beta}\,\beta\,\delta^{\emptyset}_z = 0\,. \tag{3.3}$$

Here the latter term arises only for the subgroup of dilatation (when $z = \emptyset$). Absence of obvious dependence of the Lagrangian density (3.1) from x^k yields the identity:

$$P_k L = \frac{\partial L}{\partial\psi^A} P_k\psi^A + \frac{\partial L}{\partial P_l\psi^A} P_k P_l\psi^A\,. \tag{3.4}$$

Identities (3.3) and (3.4) are 'strong' identities, which are satisfied independently of validity of equations of the field ψ^A. When these equations are fulfilled, the given identities are equivalent to existence of conservation laws. Indeed, it is possible to represent the variation of the action (3.2) as

$$\delta J = \int_\Omega (dx)\left[\frac{\delta\mathcal{L}}{\delta\psi^A}\,\bar\delta\psi^A + \frac{\partial\mathcal{L}}{\partial\beta}\,\bar\delta\beta + \partial_\mu\left(\mathcal{L}\,\overset{\circ}{h}{}^\mu{}_k\delta x^k - \overset{\circ}{h}{}^\mu{}_k\frac{\partial\mathcal{L}}{\partial P_k\psi^A}\bar\delta\psi^A\right)\right] = 0\,, \tag{3.5}$$

where the variational derivative has a standard structure. If the field equations, $\delta\mathcal{L}/\delta\psi^A = 0$, are fulfilled, the variation (3.5), with account of (2.1) and $\partial_\mu\left(\sqrt{|\overset{\circ}{g}|}\,\overset{\circ}{h}{}^\mu{}_a\right) = 0$, is equal

$$\delta J = \int_\Omega (dx)\left(\frac{\partial\mathcal{L}}{\partial\beta}\,\delta\beta + \sqrt{|\overset{\circ}{g}|}\,\overset{\circ}{h}{}^\mu{}_k\,\partial_\mu(a^l t^k_l + \omega^m M^k{}_m + \varepsilon\Delta^k)\right) = 0\,, \tag{3.6}$$

where designations for an energy–momentum tensor $t^k_{\ l}$, full $M^k_{\ m}$ and spin $S^k_{\ m}$ momemta, and also for full Δ^k and own J^k dilatation currents of the field ψ^A are introduced:

$$t^k_{\ l} = L\delta^k_l - \frac{\partial L}{\partial P_k \psi^A} P_l \psi^A \ , \tag{3.7}$$

$$M^k_{\ m} = S^k_{\ m} + I_m{}^l{}_i \, x^i \, t^k_{\ l} \ , \quad S^k_{\ m} = -\frac{\partial L}{\partial P_k \psi^A} I_m{}^A{}_B \psi^B \ , \tag{3.8}$$

$$\Delta^k = J^k - x^l \, t^k_{\ l} \ , \quad J^k = -\frac{\partial L}{\partial P_k \psi^A} w\psi^A \ . \tag{3.9}$$

Parameters a^l, ω^m and ε are constant, but arbitrary, and the area of integration Ω is arbitrary. Therefore from identical equality to zero of a variation (3.6) with the account (2.7), the following equalities follow in curvilinear system of coordinates:

$$P_k \, t^k_{\ l} = 0 \ , \qquad P_k \, M^k_{\ m} = 0 \ , \qquad \sqrt{|\overset{\circ}{g}|} \, P_k \, \Delta^k = \beta \frac{\partial \mathcal{L}}{\partial \beta} \ . \tag{3.10}$$

Equalities (3.10) are the result of the first Noether theorem. The first two equalities yield the conservation laws of the energy-momentum $t^k_{\ l}$ and the full momentum $M^k_{\ m}$ of the field ψ^A. For the conservation of the dilatation current Δ^k it is necessary that an additional condition $\partial\mathcal{L}/\partial\beta = 0$ is fulfilled as consequence of the equation of the field ψ^A (about the dilatational invariant Lagrangians with explicit dependence on the parameter β see [15], [30]).

Using the field equations, it is possible to show that the first equality (3.10) is equivalent to the identity (3.4), and the second and the third equalities (3.10) are together equivalent to the identity (3.3). Introducing designations $J^k_{\ z} = \{S^k_{\ m} \, , \, J^k \, , 0\}$ and

$$\overset{(\psi)}{\Theta}{}^k_{\ z} = J^k_{\ z} + X^l_z \, t^k_{\ l} = LX^k_z - \frac{\partial L}{\partial P_k \psi^A} (I_z{}^A{}_B \psi^B + X^l_z P_l \psi^A) \ ,$$

it is possible to replace the equalities (3.10) by one equality

$$\partial_\mu \left(\sqrt{|\overset{\circ}{g}|} \, \overset{\circ}{h}{}^\mu{}_k \, \overset{(\psi)}{\Theta}{}^k_{\ z} \right) = \beta \, \frac{\partial \mathcal{L}}{\partial \beta} \, \delta^{\varnothing}_z \ .$$

4. The principle of local invariance

We suppose now the group $\mathcal{PW}(\omega \, , \, \varepsilon \, , \, a)$ as the localized group $\mathcal{PW}(x)$, that is we consider its parameters $\{\omega^z\} = \{\omega^m \, , \, \varepsilon \, , \, a^k\}$ as arbitrary smooth enough (belonging to a class C^2) functions of coordinates $\omega^z(x)$.

Consider invariance of action integral (3.1) under $\mathcal{PW}(x)$. Assuming the quantities $\omega^z(x)$ and $\partial_\mu \omega^z(x)$ as arbitrary and independent functions of coordinates, from (3.6) we

obtain conditions $t^k_{\ l} = 0$, $M^k_{\ m} = 0$, $\Delta^k = 0$. Thus action integral (3.1) is locally invariant then and only then, when the conservations laws are valid by virtue of the identical equality to zero of the appropriate currents (3.7)–(3.9).

It is possible to avoid this physically unsatisfactory result, if in the Lagrangian density (3.1) enter some additional fields named *gauged* (or *compensating*) having that property, that the additional members, arising in action integral (3.1) owing to transformation of a field ψ^A under action of the localized group $\mathcal{PW}(x)$, will disappear by virtue of compensating them by accordingly transformed gauge fields. Therefore gauge fields should be transformed under action $\mathcal{PW}(x)$ as non-tensorial quantities extracting at this transformation terms proportional a derivative from parameters of the group $\mathcal{PW}(x)$. In case of the group $\mathcal{PW}(x)$, the new feature arises connected with the fact that in this case the variation (2.7) is equal

$$\delta g_{ij} = 2\varepsilon(x) g_{ij} \ , \tag{4.1}$$

where $\varepsilon(x)$ is an arbitrary function. Therefore the metric becomes a function of a point of spacetime and can be represented as

$$g_{ij} = \beta^2(x) g^M_{ij} \ , \tag{4.2}$$

that demands the account derivatives $P_k\beta(x)$ ($\beta(x) > 0$) in the Lagrangian.

The requirement of gauge invariance in application to the Poincaré group has been formulated in [6], [7] as some variational principle, which for a case of the localized Poincaré–Weyl group we generalize as follows.

Postulate 1 (*The principle of local invariance*). An action integral

$$J = \int_\Omega (dx)\, \mathcal{L}(\psi^A,\, P_k\psi^A,\, A^R_a,\, P_kA^R_a,\, \beta(x)\,,\, P_k\beta(x))\ , \tag{4.3}$$

where the Lagrangian density $\mathcal{L}$ describes a field ψ^A, interaction of a field ψ^A with an additional gauge field A^R_a and the free field A^R_a, is invariant under the action of the localized group $\mathcal{PW}(x)$, the gauge field being transformed as follows

$$\delta A^R_a = U^R_{za}\omega^z + S^{R\mu}_{za}\partial_\mu\omega^z \ , \tag{4.4}$$

where U and S are some matrix functions.

This variational principle allows to apply to gauge theories the first and the second Noether theorems and, in spite of a generality of the formulation, it is sufficient to determine a structure of the Lagrangian density $\mathcal{L}$ and to find the matrix functions U, S. In

the present work we generalize on the gauge theory of the localized Poincaré–Weyl group the method of construction gauge theories, developed in [15]–[17] for the Poincaré group.

The gauge fields equations, as well as the equations of the field ψ^A, are derived on the basis of a principle of stationary action, which should be chosen as the second postulate.

> **Postulate 2** (*The principle of stationary action*). The equations of the field ψ^A and the gauge fields A^R_a realize an extremum of the action integral (4.3), describing the field ψ^A, an interaction of the field ψ^A with the gauge field A^R_a and the free gauge field A^R_a.

From physical reasons it is necessary to conclude, that the full Lagrangian density $\mathcal{L}$ consists from the Lagrangian density $\mathcal{L}_0$ of free gauge fields and from the Lagrangian density $\mathcal{L}_\psi$ describing the free field ψ^A and the interaction of the field ψ^A with gauge fields. Action integrals for each of these Lagrangian densities separately need to be locally invariant, as it is natural to expect, that the gauge field can exist irrespective of the field ψ^A. We formulate the given physical requirements as the third postulate of the theory of gauge fields.

> **Postulate 3** (*An independent existence of a free gauge field*). The full Lagrangian density $\mathcal{L}$ of a physical system depends additively from the locally invariant Lagrangian density $\mathcal{L}_0$ of free gauge fields: $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\psi$, where
>
> $$\mathcal{L}_0 = \mathcal{L}_0(A^R_a\,,\, P_k A^R_a\,,\, \beta(x)\,,\, P_k\beta(x))\ , \quad \frac{\partial \mathcal{L}_0}{\partial \psi^A} = 0\ , \quad \frac{\partial \mathcal{L}_0}{\partial P_k \psi^A} = 0\ .$$

Further we shall always assume, that interaction of the field ψ^A with gauge fields is carried out only through connection with gauge fields, but not with their derivatives. In other words, all derivatives from the gauge fields are contained only in the Lagrangian density $\mathcal{L}_0$. The interaction satisfying this condition is accepted to name *minimal*. We formulate this condition as the forth postulate of the gauge fields theory.

> **Postulate 4** (*The principle of minimality of a gauge interaction*). In the Lagrangian density $\mathcal{L}_\psi$ of an interaction of a material field ψ^A with gauge fields only derivatives from the material field ψ^A are present. Thus the following conditions are satisfied,
>
> $$\frac{\partial \mathcal{L}_\psi}{\partial P_k A^R_a} = 0\ , \qquad \frac{\partial \mathcal{L}_\psi}{\partial P_k \beta} = 0\ .$$

The variation of the action integral (4.3) with regard to the variation of the area Ω reads

$$0 = \delta J = \int_\Omega (dx)\left((\partial_\mu \delta x^\mu)\,\mathcal{L} + \delta\mathcal{L}\right) = \int_\Omega (dx)\left(\frac{\delta\mathcal{L}}{\delta\psi^A}\,\bar{\delta}\psi^A + \frac{\delta\mathcal{L}}{\delta A^R_a}\,\bar{\delta}A^R_a + \frac{\delta\mathcal{L}}{\delta\beta}\,\bar{\delta}\beta\right) +$$
$$+ \int_\Omega (dx)\,\partial_\mu\left(\mathcal{L}\,\overset{\circ}{h}{}^\mu{}_k\,\delta x^k - \overset{\circ}{h}{}^\mu{}_k\frac{\partial\mathcal{L}}{\partial P_k\psi^A}\,\bar{\delta}\psi^A - \overset{\circ}{h}{}^\mu{}_k\frac{\partial\mathcal{L}}{\partial P_k A^R_a}\,\bar{\delta}A^R_a - \overset{\circ}{h}{}^\mu{}_k\frac{\partial\mathcal{L}}{\partial P_k\beta}\,\bar{\delta}\beta\right). \quad (4.5)$$

As the action of the localized subgroup of dilatation D_4 yields (4.1), the metric tensor becomes a function and is subject to a variation. The principle of stationary action will be satisfied, if the variational equations are valid,

$$\frac{\delta\mathcal{L}}{\delta\psi^A} = 0\,, \qquad \frac{\delta\mathcal{L}}{\delta A^R_a} = 0\,, \qquad \frac{\delta\mathcal{L}}{\delta\beta(x)} = 0\,. \quad (4.6)$$

It is possible to show [15] that the latter of these variational field equations is a consequence of the others. We assume that the equation of the field ψ^A is always valid. According to the Postulate 3, the full Lagrangian density of a gauge field consists from the Lagrangian density of a free gauge field and from the Lagrangian density of interaction. For the separate interaction Lagrangian density the action integral is locally invariant, but the variational equation of the gauge field A^R_a is not satisfied. The equation of the field A^R_a is valid only for the full Lagrangian density $\mathcal{L}$. In this last case on the basis of (4.5), taking into account (4.6), (4.4) and putting quantities $\omega^z(x)$, $\partial_\mu\omega^z(x)$, $\partial_\mu\partial_\nu\omega^z(x)$ as arbitrary and independent functions of coordinates, we obtain a fundamental set of identities on extremals of the fields ψ^A, A^R_a and $\beta(x)$:

$$\partial_\mu(\overset{\circ}{h}{}^\mu{}_k\,\Theta^k{}_z) = 0\,, \quad \overset{\circ}{h}{}^\mu{}_k\Theta^k{}_z - \partial_\nu\mathcal{M}^{\nu\mu}{}_z = 0\,, \quad \mathcal{M}^{(\nu\mu)}{}_z = 0\,, \quad (4.7)$$

where the following designations are introduced with regard to (2.12), (2.4) and (4.4):

$$\Theta^k{}_z = \mathcal{L}X^k_z - \frac{\partial\mathcal{L}}{\partial P_k\psi^A}(I_z{}^A{}_B\psi^B + X^l_z P_l\psi^A) -$$
$$-\frac{\partial\mathcal{L}}{\partial P_k A^R_a}(U^R_{za} + X^l_z P_l A^R_a) - \frac{\partial\mathcal{L}}{\partial P_k\beta}(\beta\delta^{\emptyset}_z + X^l_z P_l\beta)\,,$$
$$\mathcal{M}^{\nu\mu}{}_z = \overset{\circ}{h}{}^\nu{}_k\frac{\partial\mathcal{L}}{\partial P_k A^R_a}\,S^{R\mu}_{za}\,. \quad (4.8)$$

The equalities (4.7) represent the relations of the second Noether theorem written down in curvilinear system of coordinates. It is easy to be convinced that the first of these equality (representing the conservation law of the appropriate current) is a consequence of two others. Thus it is shown, that introduction of gauge fields leads to a dynamical realization of conservation laws. The quantity (4.8) represents a superpotential for the appropriate conservation current.

5. Structure of the interaction Lagrangian

Following the method developed in [15], we introduce the differential operator M_R:

$$M_R = \{M_m{}^A{}_B\,,\, M_\varnothing{}^A{}_B\,,\, M_k{}^A{}_B\}\,, \qquad R = \{m,\, \varnothing,\, k\}\,, \tag{5.1}$$

uniting the operators of full momemtum, full dilatation current and shift:

$$M_m{}^A{}_B = I_m{}^A{}_B + \delta^A_B\, \overset{\circ}{M}_m\,, \quad M_\varnothing{}^A{}_B = w\delta^A_B + \delta^A_B\, \overset{\circ}{D}\,, \quad M_k{}^A{}_B = \delta^A_B P_k\,. \tag{5.2}$$

Also let us represent the gauge field A^R_a as a set of three components:

$$A^R_a = \{A^m_a\,,\, A_a\,,\, A^k_a\}\,,$$

where A^m_a is the gauge field appropriate to the subgroup of 4-rotations (r-field), A_a is the gauge field of the subgroup of dilatation (d-field), and A^k_a is the gauge field of the subgroup of translations (t-field) of the Poincaré–Weyl group.

The following theorem on the structure of the Lagrangian density $\mathcal{L}_\psi$ of interaction of an external field with the gauge fields, which represents a generalization on the Poincaré–Weyl group $\mathcal{PW}(x)$ the appropriate theorem proved in [15] for a case of the Poincaré group.

Theorem 1. There exists a gauge field A^R_a with transformation structure (4.4) of Postulate 1 under action of the localized Poincaré–Weyl group $\mathcal{PW}(x)$ and there are such matrix functions Z, U and S of the gauge field, that the Lagrangian density

$$\mathcal{L}_\psi = \sqrt{|\,\bar{g}\,|}\, L_\psi(\psi^A,\, D_a\psi^A,\, \beta(x))\,, \quad \sqrt{|\,\bar{g}\,|} = Z\sqrt{|\overset{\circ}{g}|}\,, \tag{5.3}$$

satisfies to the principle of local invariance (Postulate 1) concerning the localized group $\mathcal{PW}(x)$, $\mathcal{L}_\psi$ being formed from the invariant concerning the non-localized group $\mathcal{PW}$ Lagrangian density $L(\psi^A\, P_k\psi^A)$ by replacement of the differential operator P_k on the gauge derivative operator

$$D_a = -A^R_a M_R\,, \tag{5.4}$$

where the operator M_R is given as (5.1). Also the following representation of the gauge t-field is valid:

$$A^k_a = D_a x^k\,. \tag{5.5}$$

Proof. Substituting in (5.4) the expression for the operator M_R, we obtain according to (5.1) and (5.2) the explicit form of a gauge derivative for the group $\mathcal{PW}(x)$:

$$D_a\psi^A = h^\mu{}_a\partial_\mu\psi^A - A^m_a I_m{}^A{}_B\psi^B - wA_a\psi^A = h^\mu{}_a D_\mu\psi^A \,, \tag{5.6}$$

$$D_\mu\psi^A = \partial_\mu\psi^A - A^m{}_\mu I_m{}^A{}_B\psi^B - wA_\mu\psi^A \,. \tag{5.7}$$

Here new quantities are introduced:

$$h^\mu{}_a = \overset{\circ}{h}{}^\mu{}_k Y^k_a \,, \quad Y^k_a = A^R_a X^k_R = A^k_a + A^m_a I_m{}^k{}_l\, x^l - A_a x^k \,, \quad Z^a_k = (Y^{-1})^a_k \,, \tag{5.8}$$

$$h^a{}_\mu = (h^{-1})^a{}_\mu = Z^a_k\, \overset{\circ}{h}{}^k{}_\mu \,, \quad A^m{}_\mu = A^m{}_a\, h^a{}_\mu \,, \quad A_\mu = A_a\, h^a{}_\mu \,. \tag{5.9}$$

By analogy to section 3, we obtain the 'strong' identities expressing conditions of invariance of the action integral for the Lagrangian density (5.3) under action of the localized group $\mathcal{PW}(x)$. The variation of action integral is equal

$$\delta\int_\Omega (dx)\,\mathcal{L}_\psi = \int_\Omega (dx)\Big(\sqrt{|\,\bar g\,|}\,(\partial_\mu\delta x^\mu)\,L_\psi + \delta\left(\sqrt{|\,\bar g\,|}\right)\,L_\psi + \sqrt{|\,\bar g\,|}\,\delta L_\psi\Big) = 0 \,. \tag{5.10}$$

We introduce a quantity $\bar g_{\mu\nu}$ (a tensor g_{ab} has values of the tensor g_{ij} (2.1)):

$$\bar g_{\mu\nu} = g_{ab}h^a{}_\mu h^b{}_\nu = g_{ab}Z^a_k Z^b_l\, \overset{\circ}{h}{}^k{}_\mu\, \overset{\circ}{h}{}^l{}_\nu \,, \quad \bar g = \det(\bar g_{\mu\nu}) = \overset{\circ}{g}\, Z^2 \,, \quad Z = \det(Z^a_k) \,, \tag{5.11}$$

and also demand that matrixes U and S in the gauge fields transformation law (4.4) were those that for the quantity g in (5.11) the following equality would be satisfied:

$$\delta\left(\sqrt{|\,\bar g\,|}\right) = -\sqrt{|\,\bar g\,|}\,(\partial_\mu\delta x^\mu) \,. \tag{5.12}$$

The proof of the existence of the quantity $g_{\mu\nu}$ with the specified property is given at the end of the section.

Then the equality (5.10) by virtue of arbitrariness of the area Ω means $\delta L_\psi = 0$:

$$\delta L_\psi = \left(\frac{\partial L_\psi}{\partial\psi^A}\right)_{D\psi=const}\delta\psi^A + \frac{\partial L_\psi}{\partial D_a\psi^A}\,\delta D_a\psi^A + \frac{\partial L_\psi}{\partial\beta}\,\delta\beta = 0 \,. \tag{5.13}$$

Using (5.6), (5.8), (4.4), let us calculate a variation $\delta D_a\psi^A$ and then substitute it and also (2.6) and (4.1) into (5.13). In the identity received we collect factors before quantities $\omega^z(x)$ and $\partial_\mu\omega^z(x)$. In view of arbitrariness of these quantities these factors separately should be equal to zero identically. The factor before $\partial_\mu\omega^z(x)$ is equal

$$\frac{\partial L_\psi}{\partial D_a\psi^A}(I_R{}^A{}_B\psi^B + X^l_R P_l\psi^A)(S^{R\mu}_{za} - \delta^R_z h^\mu{}_a) = 0 \,.$$

This equality is satisfied identically at

$$S^{R\mu}_{za} = \delta^R_z h^\mu{}_a \ . \tag{5.14}$$

Look over various sets of indexes, we find values of an unknown matrix S:

$$S^{n\mu}_{ma} = \delta^n_m h^\mu{}_a \ , \quad S^{n\mu}_{ka} = 0 \ , \quad S^{k\mu}_{ma} = 0 \ , \quad S^{l\mu}_{ka} = \delta^l_k h^\mu{}_a \ , \tag{5.15}$$

$$S^\mu_a = h^\mu{}_a \ , \quad S_{ma}{}^\mu = 0 \ , \quad S^{n\mu}{}_a = 0 \ , \quad S^{k\mu}{}_a = 0 \ , \quad S_{ka}{}^\mu = 0 \ . \tag{5.16}$$

Now we take into account that an algebraic structure of the scalar L_ψ should satisfy to the identity (3.3), which owing to the replacement of P_a by D_a is equal

$$\left(\frac{\partial L_\psi}{\partial \psi^A}\right)_{D\psi=const} I_z{}^A{}_B \psi^B + \frac{\partial L_\psi}{\partial D_a \psi^A}(I_z{}^A{}_B D_a \psi^B - I_z{}^b{}_a D_b \psi^A) + \frac{\partial L_\psi}{\partial \beta} \beta \delta^{\o}_z = 0 \ .$$

Considering the given identity, let us write out the factor at $\omega^z(x)$ in identity (5.13). As a result, we obtain some expression, which identically is equal to zero at the following set of matrixes U in the transformation law (4.4):

$$U^n_{ma} = c_m{}^n{}_q A^q_a - I_m{}^b{}_a A^n_b \ , \quad U_{ma} = -I_m{}^b{}_a A_b \ , \tag{5.17}$$

$$U^n_a = A^n_a \ , \quad U_a = A_a \ , \quad U^n_{ka} = 0 \ , \quad U_{ka} = 0 \ , \tag{5.18}$$

$$U^k_{ma} = I_m{}^k{}_l A^l_a - I_m{}^b{}_a A^k_b \ , \quad U^k_{ia} = -I_n{}^k{}_i A^n_a + \delta^k_i A_a \ , \quad U^k_a = 0 \ . \tag{5.19}$$

These expressions can be expressed in short as

$$U^R_{za} = c_z{}^R{}_Q A^Q_a - I_z{}^b{}_a A^R_b \ , \tag{5.20}$$

where each of the indexes R, Q, z can take the values of the each indexes m, k, $\o$, and the commutation relations (2.5) of the Poincaré–Weyl group $\mathcal{PW}$ should be taken into account. The expressions (5.14) and (5.20) found for an unknown function Z and unknown matrix functions U and S, for which the Lagrangian density satisfies to identity (5.10), prove the basic statements of the Theorem 1.

Now we shall prove the formula (5.5). Owing to (2.2), the quantity x^k is transformed as vector representation of the group $\mathcal{PW}$, and, comparing (2.2) and (2.6), we find, that $w[x^k] = -1$. At calculation a gauge derivative $D_a x^k$ we use the formulas (5.6), (5.8), (2.2) and (2.8):

$$D_a x^k = h^\mu{}_a \partial_\mu x^k - A^m_a I_m{}^k{}_l x^l - w[x^k] A_a x^k =$$
$$= \overset{\circ}{h}{}^\mu{}_i (A^i_a + A^m_a I_m{}^i{}_l x^l - A_a x^i) \partial_\mu x^k - A^m_a I_m{}^k{}_l x^l + A_a x^k = A^k_a \ .$$

The formula (5.5) clears up a geometrical sense of the gauge field of the subgroup of translations. This formula generalizes on the Poincaré–Weyl group the similar formula, which arises at the gauge approach for the Poincaré group [15]–[17].

Let us find transformation laws of components of the gauge field under action of the localized Poincaré–Weyl group $\mathcal{PW}(\omega\,,\,\varepsilon\,,\,a)$. The general form of the transformation law is determined on account of the principle of local invariance by the expression (4.4). We break in this expression indexes R and z on three indexes concerning subgroups of 4-rotations, dilatation and translations and substitute in these formulas the concrete values of the matrix functions U (5.17)–(5.19) and S (5.15), (5.16). As a result we obtain the general rules of transformations for the fields A^m_a, A_a and A^k_a:

$$\delta A^m_a = \omega^n c_n{}^m{}_q A^q_a - \omega^n I_n{}^b{}_a A^m_b + \varepsilon A^m_a + h^\mu{}_a\, \partial_\mu \omega^m\ , \tag{5.21}$$

$$\delta A_a = -\omega^n I_n{}^l{}_a A_l + \varepsilon A_a + h^\mu{}_a\, \partial_\mu \varepsilon\ , \tag{5.22}$$

$$\delta A^k_a = \omega^n (I_n{}^k{}_l A^l_a - I_n{}^l{}_a A^k_l) + a^l(-A^n_a I_n{}^k{}_l + A_a \delta^k_l) + h^\mu{}_a\, \partial_\mu a^k\ . \tag{5.23}$$

A variation of the quantity $h^\mu{}_a$ is determined on the basis of formulas (5.8), the transformation law[1] of the quantity Y^k_a having been determined previously on the basis of the variations (5.21)–(5.23):

$$\delta Y^k_a = -\omega^n I_n{}^l{}_a Y^k_l + \varepsilon Y^k_a + h^\mu{}_a\, \partial_\mu \delta x^k\ . \tag{5.24}$$

As a result one finds

$$\delta h^\mu{}_a = -\omega^n I_n{}^b{}_a\, h^\mu{}_b + \varepsilon h^\mu{}_a + h^\nu{}_a\, \partial_\nu \delta x^\mu\ . \tag{5.25}$$

At last, from (5.9) we find transformation rules of the quantities $h^a{}_\mu$, $A^m{}_\mu$ and A_m:

$$\delta h^a{}_\mu = \omega^n I_n{}^a{}_b\, h^b{}_\mu - \varepsilon h^a{}_\mu - h^a{}_\nu\, \partial_\mu \delta x^\nu\ , \tag{5.26}$$

$$\delta A^m{}_\mu = \omega^n c_n{}^m{}_q A^q_\mu + \partial_\mu \omega^m - A^m{}_\nu\, \partial_\mu \delta x^\nu\ ,$$

$$\delta A_\mu = \partial_\mu \varepsilon - A_\nu\, \partial_\mu \delta x^\nu\ .$$

After the geometrical interpretation of the theory (section 7) the quantity $h^\mu{}_a$ will be interpreted as a tetrad potential. We see, that at localization of the group $\mathcal{PW}$ (as well as in case of the Poincaré group [15]), the quantity $h^\mu{}_a$ is not a gauge field because it is transformed as tensor (in agree with (5.25) and (5.26)), against the gauge fields A^m_a, A_a

[1] The transformation law (5.24) is externally similar to the transformation law of gauge fields (4.4). However, parameters of this transformation δx^k do not bear in themselves (without their concrete definition) any information on the group $\mathcal{PW}(x)$. Therefore the quantity Y^k_a is impossible to consider as a gauge field for the group $\mathcal{PW}(x)$. The quantity Y^k_a is closely connected with the tetrads $h^\mu{}_a$.

and A^k_a, which are transformed concerning the group $\mathcal{PW}(x)$ by non-tensorial rules. We note, that in [11], [13] it was specified the fact, that tetrads are not true potentials of a gravitational field.

We shall prove now validity of the formula (5.12) for components of the quantity $g_{\mu\nu}$ determined by (5.11). Determined by formulas (5.15)–(5.19), the expressions for matrix functions S and U yield the transformation law (5.26) for the quantity $h^a{}_\mu$. Owing to the definition (5.11), we have

$$\delta\left(\sqrt{\mid \bar{g} \mid}\right) = \frac{1}{2\sqrt{\mid \bar{g} \mid}}\,\delta \mid \bar{g} \mid = \frac{1}{2\sqrt{\mid \bar{g} \mid}} \mid \bar{g} \mid \; g^{\mu\nu}\delta g_{\mu\nu} = \frac{1}{2}\sqrt{\mid \bar{g} \mid}\, g^{\mu\nu}(h^a{}_\mu h^b{}_\nu \delta g_{ab} + 2g_{ab}h^b{}_\nu \delta h^a{}_\mu)\ .$$

Substituting here the expressions for variations of the quantities g_{ab} and $h^a{}_\mu$ from (2.7) and (5.26), we find

$$\delta\left(\sqrt{\mid \bar{g} \mid}\right) = \frac{1}{2}\sqrt{\mid \bar{g} \mid}\, g^{\mu\nu}\Big(2\varepsilon(x) g_{ab}h^a{}_\mu h^b{}_\nu + 2g_{ab}h^b{}_\nu\,(\omega^n(x) I_n{}^a{}_c\, h^c{}_\mu - \varepsilon(x) h^a{}_\mu -$$
$$- h^a{}_\sigma\,\partial_\mu \delta x^\sigma)\Big) = \frac{1}{2}\sqrt{\mid \bar{g} \mid}\,(8\varepsilon(x) + \omega^n(x) I_n{}^a{}_a - 8\varepsilon(x) - 2\partial_\mu \delta x^\mu) = -\sqrt{\mid \bar{g} \mid}\,\partial_\mu \delta x^\mu\ .$$

Thus it is proved, that such matrix functions S and U exist, namely, given by the expressions (5.15)–(5.19), for which the equality (5.12) is satisfied. $\sqcup$

6. Free gauge field Lagrangian. Equations of gauge fields

For definition of structure of the free gauge field Lagrangian for the group $\mathcal{PW}(x)$ we need to find such functions of the gauge fields, which are transformed concerning the group $\mathcal{PW}(x)$ as tensors. Knowing, that the gauge derivative $D_a\psi^A$ is tensor, we calculate the commutator of gauge derivatives of a field ψ^A:

$$2D_{[a}D_{b]}\psi^A = -F^m{}_{ab} I_m{}^A{}_B \psi^B + wF_{ab}\psi^A - F^c{}_{ab} D_c \psi^A\ .$$

The quantities $F^m{}_{ab}$, F_{ab} and $F^c{}_{ab}$ are tensors. They are defined by expressions

$$F^m{}_{ab} = 2h^\lambda{}_{[a}\partial_{|\lambda|} A^m_{b]} + A^m_c C^c{}_{ab} - c_n{}^m{}_q A^n_a A^q_b\ , \tag{6.1}$$

$$F_{ab} = 2h^\lambda{}_{[a}\partial_{|\lambda|} A_{b]} + A_c C^c{}_{ab}\ , \tag{6.2}$$

$$F^c{}_{ab} = C^c{}_{ab} + 2I_n{}^c{}_{[a} A^n_{b]} + 2A_{[a}\delta^c_{b]}\ , \tag{6.3}$$

$$C^c{}_{ab} = -2h^c{}_\tau h^\lambda{}_{[a}\partial_{|\lambda|} h^\tau{}_{b]} = 2h^\lambda{}_a h^\tau{}_b \partial_{[\lambda} h^c{}_{\tau]}\ .$$

Besides, the following contraction of a gauge derivative is also a vector :

$$Q_a = -g^{bc} D_a g_{bc} = -h^\mu{}_a \, g^{bc} \partial_\mu g_{bc} - 2 g^{bc} A^z_a I_{zbc} = 8(A_a - h^\mu{}_a \, \partial_\mu \ln \beta(x)) \; . \tag{6.4}$$

With the help of the variations of the gauge fields (5.21)–(5.23) and also of the variation of the tetrads (5.25), (5.26), it is possible to show by explicit calculations that (6.1)–(6.3), (6.4) are transformed as covariant quantities under action of the group $\mathcal{PW}(x)$:

$$\begin{aligned}
\delta F^m{}_{ab} &= \omega^n (c_n{}^m{}_q F^q{}_{ab} - I_n{}^c{}_a F^m{}_{cb} - I_n{}^c{}_b F^m{}_{ac}) + 2\varepsilon F^m{}_{ab} \; , \\
\delta F_{ab} &= -\omega^n (I_n{}^c{}_a F_{cb} + I_n{}^c{}_b F_{ac}) + 2\varepsilon F_{ab} \; , \\
\delta F^c{}_{ab} &= \omega^n (I_n{}^c{}_d F^d{}_{ab} - I_n{}^d{}_a F^c{}_{db} - I_n{}^d{}_b F^c{}_{ad}) + \varepsilon F^c{}_{ab} \; , \\
\delta Q_a &= -\,\omega^n I_n{}^b{}_a Q_b + \varepsilon Q_a \; .
\end{aligned}$$

These tensor quantities contain derivatives from gauge fields only of the first degree, therefore it is natural to name them as gauge field strengthes. For construction of the free gauge field Lagrangian density it is necessary to use the scalars formed from the gauge field strengthes. As a result we come to a conclusion about the structure of the free gauge field Lagrangian density.

Theorem 2. The following Lagrangian density

$$\mathcal{L}_0 = \sqrt{|\,\bar g\,|}\, L_0(F^m{}_{ab},\, F_{ab},\, F^c{}_{ab},\, Q_a,\, \beta(x)) \; , \tag{6.5}$$

where L_0 is a scalar function formed from the gauge field strengthes (6.1)–(6.3), (6.4), satisfies to the principle of local invariance.

A full Lagrangian density of a set of a field ψ^A and a gauge field is

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\psi \; , \tag{6.6}$$

where $\mathcal{L}_0$ is given by the expression (6.5), and $\mathcal{L}_\psi$ – by the expression (5.3). For the full Lagrangian density (6.6) the variational gauge field equations (4.6) are satisfied:

$$\frac{\delta \mathcal{L}_0}{\delta A^m_a} = -\frac{\partial \mathcal{L}_\psi}{\partial A^m_a} \, , \quad \frac{\delta \mathcal{L}_0}{\delta A_a} = -\frac{\partial \mathcal{L}_\psi}{\partial A_a} \, , \quad \frac{\delta \mathcal{L}_0}{\delta A^k_a} = -\frac{\partial \mathcal{L}_\psi}{\partial A^k_a} \, , \quad \frac{\delta \mathcal{L}_0}{\delta \beta(x)} = -\frac{\delta \mathcal{L}_\psi}{\delta \beta(x)} \; . \tag{6.7}$$

As it has been already pointed out, the latter of these variational field equations is a consequence of the others. The right parts of these equations represent sources of the gauge fields, which are calculated with the help of expression (5.3):

$$-\frac{\partial \mathcal{L}_\psi}{\partial A^m_a} = \sqrt{|\,\bar g\,|}\, (\overset{\circ}{M}{}^a{}_m + S^a{}_m) \; ,$$

$$
\overset{\circ}{M}{}^{a}{}_{m} = I_{m}{}^{c}{}_{b}\, x^{b} \overset{(g)}{t}{}^{a}{}_{c} \, , \quad \sqrt{|\, \bar{g}\, |}\, S^{a}{}_{m} = \frac{\partial \mathcal{L}_{\psi}}{\partial D_{a}\psi^{A}} I_{m}{}^{A}{}_{B}\psi^{B} \, , \tag{6.8}
$$

$$
-\frac{\partial \mathcal{L}_{\psi}}{\partial A_{a}} = \sqrt{|\, \bar{g}\, |}\, \Delta^{a} \, , \ \ \Delta^{a} = -x^{b} \overset{(g)}{t}{}^{a}{}_{b} + J^{a} \, , \ \ \sqrt{|\, \bar{g}\, |}\, J^{a} = \frac{\partial \mathcal{L}_{\psi}}{\partial D_{a}\psi^{A}} w\psi^{A} \, , \tag{6.9}
$$

$$
-\frac{\partial \mathcal{L}_{\psi}}{\partial A^{k}_{a}} = \sqrt{|\, \bar{g}\, |} \ \overset{(g)}{t}{}^{a}{}_{k} \, , \quad \sqrt{|\, \bar{g}\, |} \ \overset{(g)}{t}{}^{a}{}_{k} = Z^{a}_{l} \left(\mathcal{L}_{\psi}\delta^{l}_{k} - \frac{\partial \mathcal{L}_{\psi}}{P_{l}\psi^{A}} \, P_{k}\psi^{A} \right) \, . \tag{6.10}
$$

Here $\overset{(g)}{t}{}^{a}{}_{k}$ is an energy-momentum tensor, $S^{a}{}_{m}$ is a spin, $\overset{\circ}{M}{}^{a}{}_{m}$ is an orbital momenta of an external field ψ^{A} [31], Δ^{a} is full, and J^{a} is a proper dilatation currents of ψ^{A} [32].

The equations (6.7)–(6.10) allow to generalize on the localized group $\mathcal{PW}(x)$ the theorem on the sources of gauge fields, proved in [6], [15] for the localized Poincaré group.

> **Theorem 3** (*on the sources of gauge fields*). Sources of the gauge fields introduced by the localized Poincaré–Weyl group $\mathcal{PW}(x)$, are invariants of the Noether theorem appropriate to the non-localized Poincaré–Weyl group $\mathcal{PW}$.

According to this theorem, the dilatation current Δ^{a} of an external field (connected with possible presence of a dilatation charge in nature [33]–[35]) arises as a source of the gauge field in addition to the currents introduced by the localized Poincaré group.

As an example let us consider a case, when an external field is a spinor field. Then, according to the Theorem 1, a spinor field Lagrangian interacting with the gauge fields of the Poincaré–Weyl group reads

$$
\mathcal{L}_{\psi} = \sqrt{|\, \bar{g}\, |}\, L_{\psi} \, , \quad L_{\psi} = \bar{\psi}_{A}\gamma^{a} D_{a}\psi^{A} - m\, \bar{\psi}_{A}\psi^{A} \, ,
$$

where the gauge derivative D_{a} is determined by expressions (5.6) and (5.8). We consider the gauge fields as weak fields and pass to a Cartesian system of coordinates, for which $\overset{\circ}{h}{}^{\mu}{}_{a} = \delta^{\mu}_{a}$. Substituting the expressions for D_{a} in the Lagrangian, we receive

$$
L_{\psi} = \bar{\psi}_{A}\gamma^{a} D^{(t)}_{a}\psi^{A} - m\, \bar{\psi}_{A}\psi^{A} + L_{\psi M} + L_{\psi D} \, , \tag{6.11}
$$

$$
D^{(t)}_{a}\psi^{A} = A^{k}_{a}\partial_{k}\psi^{A} - A^{m}_{a} I_{m}{}^{A}{}_{B}\psi^{B} - wA_{a}\psi^{A} \, , \tag{6.12}
$$

$$
L_{\psi M} = A^{m}_{a} I_{m}{}^{k}{}_{l}\, x^{l} \bar{\psi}_{A}\gamma^{a}\partial_{k}\psi^{A} \, , \qquad L_{\psi D} = -A_{a}x^{k}\bar{\psi}_{A}\gamma^{a}\partial_{k}\psi^{A} \, . \tag{6.13}
$$

Here the first term contains a derivative $D^{(t)}_{a}\psi^{A}$, which appears from (5.6) by exchange the tetrads $h^{\mu}{}_{a}$ for the translation gauge field A^{k}_{a}. This term contains a direct interaction of a gravitational field with the spin momentum of ψ^{A}, which is standard for General Relativity. The second term describes a direct interaction of the Lorentz gauge field with

the orbital momentum of the spinor field. The third term describes a direct interaction of the dilatation gauge field with the orbital dilatation current. Both these interactions are absent in all those theories, in which the tetrads do not represent as (5.8), in particular, in the theories advanced in [24], [25] and also in General Relativity.

Let us calculate the second term for example. To that end we calculate the expression (6.10) for the energy-momentum tensor of the spinor field:

$$t^a{}_k = Z^a_k\, L_\psi - \bar{\psi}_A \gamma^a \partial_k \psi^A \ ,$$

and also use the expression for generators of vector representation of the group $\mathcal{PW}(x)$:

$$I_{ij}{}^a{}_b = \delta^a_i g_{jb} - \delta^a_j g_{ib} \quad (m \to \{i, j\},\ i < j)\ . \tag{6.14}$$

As a result we obtain:

$$L_{\psi M} = (1/2) A^{ij}_a M^a_{ij}\ , \qquad M^a_{ij} = x_i\, t^a{}_j - x_j\, t^a{}_i\ ,$$

where it is taken into account that $L_\psi = 0$ by virtue of the spinor field equation.

The given interaction describes the effect, which will become apparent, in particular, in a precession of electronic orbits under action of an external gravitational field and theoretically can be used for detecting gravitational waves.

7. Interactions of gauge fields

Let's consider a full group of gauge symmetries $\Gamma(x)$, into which the group $\mathcal{PW}(x)$ enters as a component of a direct product. As an example, we shall consider a group

$$\Gamma(x) = \mathcal{PW}(x) \bigotimes SU_3(x) \bigotimes U_1(x)\ ,$$

where $SU_3(x)$ is the non-Abelian gauge color group of quantum chromodynamics, and $U_1(x)$ is the gauge group of electrodynamics. Then, applying to the group $\Gamma(x)$ the general theory of gauge fields [15], we obtain, according to the Theorem 2, that a strength tensor of a unified gauge field reads

$$F^M{}_{ab} = 2h^\lambda{}_{[a} \partial_{|\lambda|} A^M_{b]} + A^M_c C^c{}_{ab} - c_N{}^M{}_Q A^N_a A^Q_b\ , \tag{7.1}$$

where indexes M, N, Q run the values of indexes of all infinitesimal operators of all components of the direct product. Further it is necessary to take into account, that the various infinitesimal operators of components of the direct product commute. As a result,

structural constants and a metric g_{MN} of a group space, and also squares of the tensor (7.1) break up to the blocks concerning everyone of a component of the direct product.

Then the gauge field strength tensor (7.1) will be represented as a set of components $F^M{}_{ab} = \{F^m{}_{ab},\ F^i{}_{ab},\ F_{ab}\}$. Here a tensor $F^m{}_{ab}$ concerns to the group $\mathcal{PW}(x)$ and is given by the expression (6.1). A tensor $F^i{}_{ab}$ describes a color gauge field, and a tensor F_{ab} describes an electromagnetic field. These tensors are given by expressions

$$F^i{}_{ab} = 2h^\lambda{}_{[a}\partial_{|\lambda|}A^i_{b]} + A^i_c C^c{}_{ab} - c_k{}^i{}_l A^k_a A^l_b\ , \qquad F_{ab} = 2h^\lambda{}_{[a}\partial_{|\lambda|}A_{b]} + A_c C^c{}_{ab}\ , \tag{7.2}$$

where $c_k{}^i{}_l$ are structural constants of the group SU_3. Formulas (7.2) describe interaction of color and electromagnetic fields with the gauge field of the group $\mathcal{PW}(x)$.

In connection with stated, it is erroneous to describe the given interactions by the replacement of usual derivatives on gauge covariant ones:

$$F^i{}_{ab} = 2D_{[a}A^i_{b]} + A^i_c C^c{}_{ab} - c_k{}^i{}_l A^k_a A^l_b\ , \qquad F_{ab} = 2D_{[a}A_{b]} + A_c C^c{}_{ab}\ , \tag{7.3}$$

that it is often accepted to use.

8. Geometrical interpretation

Well-known, that the theory of gauge fields can be interpreted in terms of differential geometry and of fiber bundles (see [1], [11]–[13], [36], [37] and the literature cited there). Already in the paper of Weyl [19] a generalization of Riemann geometry to Weyl geometry was found as a consequence of the requirement of invariance of the theory concerning local change of scales. A fiber bundles treatment of Weyl geometry one can found in [38].

In [5]–[9], [15] it was shown, how as a consequence of localization of the Poincaré group the Riemann–Cartan geometry arises. Let us show that the results of the previous sections can be interpreted as a realization the Weyl–Cartan differential geometry on the spacetime manyfold $\mathcal{M}$. In a basis of the given geometrical interpretation there contains the identification of the gauge derivative (5.7) and a covariant derivative on the spacetime manyfold, and also the interpretation of a frame $\vec{e}_a$ as an orthogonal frame of the space (tangent to the manyfold $\mathcal{M}$), in which the localized Poincaré–Weyl group operates:

$$\vec{e}_a = \vec{e}_\mu h^\mu{}_a\ , \qquad \{\vec{e}_\mu\} = \{\partial_\mu\}\ , \qquad [\vec{e}_a,\ \vec{e}_b] = -C^c{}_{ab}\vec{e}_c\ , \quad C^c{}_{ab} = 2h^\lambda{}_a h^\tau{}_b \partial_{[\lambda}h^c{}_{\tau]}\ . \tag{8.1}$$

Thus the basis $\vec{e}_a$ is a non-holonomic basis, and a quantity $C^c{}_{ab}$ is an object of nonholonomity [29]. The shift operator P_a is redefined: $P_a = h^\mu{}_a\partial_\mu$, $[P_a,\ P_b] = -C^c{}_{ab}P_c$, and

represents the shift operator of a tangent space. The metric tensor g_{ab} appears to be a metric tensor of a tangent space, which element of length is equal $dx^a = h^a{}_\mu dx^\mu$. Then the square of an element of length of this space will be equal

$$ds^2 = g_{ab} dx^a dx^b = g_{ab} h^a{}_\mu h^b{}_\nu dx^\mu dx^\nu = \bar{g}_{\mu\nu} dx^\mu dx^\nu \ ,$$

that allows to interpret quantities $\bar{g}_{\mu\nu}$, calculated under the formula (5.11), as components of a Weyl–Cartan metric tensor of spacetime manifold $\mathcal{M}$ in coordinate holonomic basis:

$$\bar{g}_{\mu\nu} = \breve{g}(\vec{e}_\mu, \vec{e}_\nu) = \breve{g}(\vec{e}_a, \vec{e}_b)\, h^a{}_\mu h^b{}_\nu = g_{ab} h^a{}_\mu h^b{}_\nu = \beta^2\, g_{\mu\nu} \ , \quad g_{\mu\nu} = g^M_{ab} h^a{}_\mu h^b{}_\nu \ . \tag{8.2}$$

$$\sqrt{|\, \bar{g}\,|} = \beta^4 \sqrt{|\, g\,|} = \beta^4\, h \ , \quad g = \det\left(g_{\mu\nu}\right) \ , \quad h = \det\left(h^a{}_\mu\right) \ . \tag{8.3}$$

At such interpretation of the quantities $\bar{g}_{\mu\nu}$, the formula (5.12) becomes obvious. The quantities $g_{\mu\nu}$ are the coordinate holonomic components of a Riemann–Cartan metric tensor of spacetime manifold $\mathcal{M}$. Two types of indexes, arising in the theory – tetrad $a,\ b,\ \ldots$ and coordinate $\mu,\ \nu,\ \ldots$, change each other by means of the quantities $h^a{}_\mu$, which are interpreted as tetrads. It is generally conventional that contractions of quantities with Greek holonomic indexes are performed with the Riemann–Cartan metric tensor $g_{\mu\nu}$.

In the formula for a gauge derivative (5.7) it is necessary to use the expression (6.14) for generators $I_m{}^A{}_B$ of a vector representation $\psi^A = v^a$ of the Poincaré–Weyl group, a weight of a vector field being equal $w[v^a] = -1$. Then, equating the expression for a gauge derivative from a vector:

$$D_\mu v^a = \partial_\mu v^a - A^m{}_\mu I_m{}^a{}_b v^b + A_\mu v^a \ ,$$

to the expression for the covariant derivative of a vector in differential geometry: $\nabla_\mu v^a = \partial_\mu v^a + \Gamma^a{}_{b\mu} v^b$, we find connection coefficients in non-holonomic basis:

$$\Gamma^a{}_{b\mu} = -A^m{}_\mu I_m{}^a{}_b + \delta^a_b A_\mu \ . \tag{8.4}$$

With the purpose to define a covariant derivative ∇_μ for quantities with coordinate indexes, it is postulated, that

$$\nabla_\lambda h^a{}_\mu = \partial_\lambda h^a{}_\mu + \Gamma^a{}_{b\lambda} h^b{}_\mu - \Gamma^\nu{}_{\mu\lambda} h^a{}_\nu = 0 \ .$$

From this formula we find connection coefficients in holonomic coordinate basis:

$$\Gamma^\lambda{}_{\nu\mu} = h^\lambda{}_a h^b{}_\nu \Gamma^a{}_{b\mu} + h^\lambda{}_a \partial_\mu h^a{}_\nu \ . \tag{8.5}$$

According to (4.1), under action of the group $\mathcal{PW}(x)$ the metric tensor of a tangent space is multiplied by an arbitrary function and can be represented as (4.2) [39]–[41].

Calculating a variation of this expression (4.2) and comparing to a variation of the metric tensor (4.1), we find that $\delta\beta = \beta\varepsilon(x)$. Thus the field $\beta(x)$ has the weight $w[\beta(x)] = 1$. This field coincides with the scalar field introduced by Dirac [26], and can be represented as $\beta(x) = \exp\sigma(x)$, where $\sigma(x)$ is a *dilaton* field. The field $\beta(x)$ is also similar to the 'measure' scalar field introduced by Utiyama [27]. In essence the field $\beta(x)$ is a factor of components of the tangent space metric tensor in Weyl–Cartan geometry.

It is known from differential geometry [29] that a nonmetricity tensor is equal

$$Q_{ab\mu} = -\nabla_\mu g_{ab} = -\partial_\mu g_{ab} + 2\Gamma_{(ab)\mu} \ .$$

Substituting here and also in (6.4) the expression (4.2), we obtain

$$Q_{ab\mu} = \frac{1}{4} g_{ab} Q_\mu \ , \quad Q_\mu = g^{ab} Q_{ab\mu} \ , \quad Q_\mu = 8(A_\mu - \partial_\mu \ln\beta(x)) = Q_a h^a{}_\mu \ . \tag{8.6}$$

If the nonmetricity tensor satisfies to first two equalities (8.6) then nonmetricity is Weyl's nonmetricity. In this case a trace Q_μ of the nonmetricity tensor is named Weyl vector, it is expressed through the vector (6.4).

Let us introduce quantities

$$F^m{}_{\mu\nu} = F^m{}_{ab} h^a{}_\mu h^b{}_\nu = 2\partial_{[\mu} A^m{}_{\nu]} - c_n{}^m{}_q A^n{}_\mu A^q_\nu \ , \tag{8.7}$$

$$F_{\mu\nu} = F_{ab} h^a{}_\mu h^b{}_\nu = 2\partial_{[\mu} A_{\nu]} \ , \tag{8.8}$$

$$F^c{}_{\mu\nu} = F^c{}_{ab} h^a{}_\mu h^b{}_\nu = 2\partial_{[\mu} h^c{}_{\nu]} + 2 I_n{}^c{}_a h^a{}_{[\mu} A^n_{\nu]} + 2 A_{[\mu} h^c_{\nu]} \ , \tag{8.9}$$

which represent (together with (8.6)) the gauge field strengthes for a new set of dynamic variables $\{A^m{}_\mu,\, A_\mu,\, h^a{}_\mu,\, \beta(x)\}$.

Substituting in the curvature of spacetime manyfold the expression for connection coefficients (8.4) and using the commutation relations of the generators of the Lorentz subgroup and also (8.7), (8.8), we obtain a representation for the curvature tensor appropriate to known decomposition of the Weyl–Cartan curvature tensor on symmetric and antisymmetric parts [29]:

$$\bar{R}^a{}_{b\mu\nu} = 2\partial_{[\mu} \Gamma^a{}_{|b|\nu]} + 2\Gamma^a{}_{c[\mu} \Gamma^c{}_{|b|\nu]} = -I_m{}^a{}_b F^m{}_{\mu\nu} + \delta^a_b F_{\mu\nu} \ . \tag{8.10}$$

Using (8.5) and taking into account (8.4) and (8.9), we obtain the expression for a torsion tensor of spacetime manyfold:

$$T^\lambda{}_{\mu\nu} = 2\Gamma^\lambda{}_{[\nu\mu]} = 2h^\lambda{}_a h^b{}_{[\nu} \Gamma^a{}_{|b|\mu]} + 2h^\lambda{}_a \partial_{[\mu} h^a{}_{\nu]} = h^\lambda{}_a F^a{}_{\mu\nu} \ . \tag{8.11}$$

The relations (8.1), (8.4), (8.6), (8.10), (8.11) establish the connection between the geometrical quantities of spacetime manifold and the relations of the gauge fields theory for the localized Poincaré–Weyl group.

It is possible to obtain field equations by variation of the action integral for the general Lagrangian density (6.6) with respect to dynamical variables $\{A^m{}_\mu,\ A_\mu,\ h^a{}_\mu,\ \beta(x)\}$:

$$\frac{\delta\mathcal{L}}{\delta A^m{}_\mu}=0\ ,\qquad \frac{\delta\mathcal{L}}{\delta A_\mu}=0\ ,\qquad \frac{\delta\mathcal{L}}{\delta h^a{}_\mu}=0\ ,\qquad \frac{\delta\mathcal{L}}{\delta\beta(x)}=0\ . \tag{8.12}$$

As it has been already pointed out, the last field equation is a consequence of the others. The first of these field equations (8.12) can be represented as

$$\partial_\nu\frac{\partial\mathcal{L}_0}{\partial F^m{}_{\mu\nu}}=\frac{1}{2}\sqrt{|\,\bar g\,|}\,(S^\mu_{(0)m}+S^\mu{}_m)\ , \tag{8.13}$$

$$\sqrt{|\,\bar g\,|}\,S^\mu{}_m=-\frac{\partial\mathcal{L}_\psi}{\partial A^m{}_\mu}=\frac{\partial\mathcal{L}_\psi}{\partial D_\mu\psi^A}I_m{}^A{}_B\psi^B\ ,$$

$$\sqrt{|\,\bar g\,|}\,S^\mu_{(0)m}=-\frac{\partial\mathcal{L}_0}{\partial A^m{}_\mu}=2\frac{\partial\mathcal{L}_0}{\partial F^n{}_{\mu\nu}}c_m{}^n{}_qA^q{}_\nu+2\frac{\partial\mathcal{L}_0}{\partial F^c{}_{\mu\nu}}I_m{}^c{}_a\,h^a{}_\nu\ . \tag{8.14}$$

The second of these field equations (8.12) can be written down as follows

$$\partial_\nu\frac{\partial\mathcal{L}_0}{\partial F_{\mu\nu}}=\frac{1}{2}\sqrt{|\,\bar g\,|}\,(J^\mu_{(0)}+J^\mu)\ ,\quad \sqrt{|\,\bar g\,|}\,J^\mu=-\frac{\partial\mathcal{L}_\psi}{\partial A_\mu}=\frac{\partial\mathcal{L}_\psi}{\partial D_\mu\psi^A}w\psi^A\ , \tag{8.15}$$

$$\sqrt{|\,\bar g\,|}\,J^\mu_{(0)}=-\frac{\partial\mathcal{L}_0}{\partial A_\mu}=-2\frac{\partial\mathcal{L}_0}{\partial F^a{}_{\mu\nu}}h^a{}_\nu-8\frac{\partial\mathcal{L}_0}{\partial Q_\mu}\ . \tag{8.16}$$

Third of these field equations (8.12) can be represented as:

$$\partial_\nu\frac{\partial\mathcal{L}_0}{\partial F^a{}_{\mu\nu}}=-\frac{1}{2}\sqrt{|\,\bar g\,|}\,(t^\mu_{(0)a}+t^\mu_{(\psi)a})\ , \tag{8.17}$$

$$\sqrt{|\,\bar g\,|}\,t^\mu_{(\psi)a}=\frac{\partial\mathcal{L}_\psi}{\partial h^a{}_\mu}=h^\mu{}_a\mathcal{L}_\psi-\frac{\partial\mathcal{L}_\psi}{\partial D_\mu\psi^A}D_a\psi^A\ , \tag{8.18}$$

$$\sqrt{|\,\bar g\,|}\,t^\mu_{(0)a}=\frac{\partial\mathcal{L}_0}{\partial h^a{}_\mu}=h^\mu{}_a\mathcal{L}_0-2F^m{}_{a\nu}\frac{\partial\mathcal{L}_0}{\partial F^m{}_{\mu\nu}}-2F_{a\nu}\frac{\partial\mathcal{L}_0}{\partial F_{\mu\nu}}-$$
$$-2F^b{}_{a\nu}\frac{\partial\mathcal{L}_0}{\partial F^b{}_{\mu\nu}}+2A^m{}_\nu I_m{}^b{}_a\frac{\partial\mathcal{L}_0}{\partial F^b{}_{\mu\nu}}-2A_\nu\frac{\partial\mathcal{L}_0}{\partial F^a{}_{\mu\nu}}-Q_a\frac{\partial\mathcal{L}_0}{\partial Q_\mu}\ . \tag{8.19}$$

The quantity (8.18) represents the known expression for the canonical energy-momentum of an external field [9].

The quantities (8.14), (8.16) and (8.19) represent, accordingly, an internal spin momentum, a proper dilatation current and an energy-momentum of free gauge fields. These currents are conserved in the sum with the appropriate currents of an external field ψ^A.

The given conservation laws are simple consequences of the gauge field equations (8.13), (8.15) and (8.17):

$$\partial_\mu \left(\sqrt{|\,\bar g\,|}\,(S^\mu_{(0)m} + S^\mu{}_m) \right) = 0 \,, \quad \partial_\mu \left(\sqrt{|\,\bar g\,|}\,(J^\mu_{(0)} + J^\mu) \right) = 0 \,,$$
$$\partial_\mu \left(\sqrt{|\,\bar g\,|}\,(t^\mu_{(0)a} + t^\mu_{(\psi)a}) \right) = 0 \,.$$

The field equations (8.13), (8.15) and (8.17) can be represented in a geometrical form with the help of a notion of the gauge derivative (5.7), which however does not operate on the Greek coordinate indexes:

$$D_\nu \left(\frac{\partial \mathcal{L}_0}{\partial F^m{}_{\mu\nu}} \right) = \frac{1}{2}\sqrt{|\,\bar g\,|}\,S^\mu{}_m + \frac{\partial \mathcal{L}_0}{\partial F^b{}_{\mu\nu}} I_m{}^b{}_a h^a{}_\nu \,, \tag{8.20}$$

$$D_\nu \left(\frac{\partial \mathcal{L}_0}{\partial F_{\mu\nu}} \right) = \frac{1}{2}\sqrt{|\,\bar g\,|}\,J^\mu - \frac{\partial \mathcal{L}_0}{\partial F^a{}_{\mu\nu}} h^a{}_\nu - 4\frac{\partial \mathcal{L}_0}{\partial Q_\mu} \,, \tag{8.21}$$

$$D_\nu \left(\frac{\partial \mathcal{L}_0}{\partial F^a{}_{\mu\nu}} \right) - F^m{}_{a\nu}\frac{\partial \mathcal{L}_0}{\partial F^m{}_{\mu\nu}} - F^b{}_{a\nu}\frac{\partial \mathcal{L}_0}{\partial F^b{}_{\mu\nu}} -$$
$$-F_{a\nu}\frac{\partial \mathcal{L}_0}{\partial F_{\mu\nu}} - \frac{1}{2}Q_a\frac{\partial \mathcal{L}_0}{\partial Q_\mu} + \frac{1}{2}h^\mu{}_a\mathcal{L}_0 = -\frac{1}{2}\sqrt{|\,\bar g\,|}\,t^\mu{}_{(\psi)a} \,. \tag{8.22}$$

These equations, the latter of which generalizes the Einstein equation to arbitrary Lagrangian, generalize the field equations for the Poincaré group [15], [17].

By explicit calculations it is possible to prove the following theorem generalizing and specifying the theorem, given in [15] for the Poincaré group.

Theorem 4. The gauge field equations (6.7) obtained according to the principle of stationary action for a set of fields $\{A^m_a,\, A_a,\, A^k_a,\, \beta(x)\}$ are satisfied, if and only if the variational field equations (8.12) are valid for a set of fields $\{A^m{}_\mu,\, A_\mu,\, h^a{}_\mu,\, \beta(x))\}$, provided that the tetrads $h^a{}_\mu$ are represented by the formulas (5.8)–(5.9).

The given theorem shows that in those cases, in which the tetrads structure is insignificant, the various choice of dynamic variables specified in the theorem 4, results in dynamically equivalent theories. However, in some cases, for example, in a quantum theory of gravitational field, in which use of true gravitational potentials is important, the given tetrad structure can appear essential. In this case the choice as the dynamic variables the quantities $\{A^m_a,\, A_a,\, A^k_a,\, \beta(x)\}$ results in the theory, richer by the opportunities.

As analogy it is possible to point out a connection between metric and tetrad formulations of a gravitational field theory. The tetrad theory of gravitation, though results to the formulation equivalent for non-spinor matter to the metric theory of gravitation with

respect to field dynamical equations, nevertheless is richer by the opportunities realized at the spinor description in Riemann geometry.

9. Conclusion

On the basis of the general principles of the gauge fields theory the gauge theory for the Poincaré–Weyl group was constructed. The abstract expression for the gauge derivative was obtained: $D_a = -A_a^R M_R$. It was shown that, as against [24], [25], the tetrads are not true gauge fields, but represent some sufficiently complex functions from gauge fields: Lorenzian A_a^m, translational A_a^k and dilatational A_a, the relation $A_a^k = D_a x^k$ being valid. It is possible to expect that the knowledge of the true gauge potentials of a gravitational field appears essential at construction of the quantum theory of gravity.

The gauge fields equations, which sources are an energy-momentum tensor, spin and orbital momenta, and also a dilatation current of an external field are obtained. A new effect of the direct interaction of the Lorenzian gauge field with the orbital momentum of an external field appears. A geometrical interpretation of the theory is developed and it is shown that as a result of localization of the Poincaré–Weyl group spacetime becomes a Weyl–Cartan space. Also the geometrical interpretation of the Dirac's scalar field β [26]) (and thereby of a dilaton field and also of the Utiyama measure scalar field [27]) as a component of the metric tensor of a tangent space in Weyl–Cartan geometry was obtained. This field is necessary in construction a field theory in a Weyl–Cartan space [42], [43].

The gauge invariant Lagrangian of the proper gauge fields is an arbitrary scalar function from the gauge strengthes of the theory containing derivatives from the gauge fields not higher than the first order:

$$\mathcal{L}_0 = \sqrt{|\,\bar{g}\,|}\,L_0(F^m{}_{ab},\, F_{ab},\, F^c{}_{ab}, Q_a,\, \beta(x))\ , \qquad \sqrt{|\,\bar{g}\,|} = \beta^4\, h\ , \quad h = \det\,(h^a{}_\mu)\ .$$

The most simple Lagrangian of such kind, allowing to all gauge fields to be realized dynamically, can be constructed as

$$\begin{gathered}
\mathcal{L}_0 = 2hf_0\beta^4(I_{m\,b}^{\,a}F^{m\,b}{}_{a} + fF^m{}_{ab}F_m{}^{ab} + \rho F^c{}_{ab}F_c^{ab} + \lambda F_{ab}F^{ab} + \xi Q_a Q^a + \zeta F^c{}_{ac}Q^a + \Lambda) = \\
2hf_0\left(\frac{1}{2}\beta^2\bar{R} + 2f\bar{R}_{[ab]\mu\nu}\bar{R}^{[ab]\mu\nu} + 4\lambda(\partial_{[\mu}A_{\nu]})^2 + \beta^2(\rho T_{\lambda\mu\nu}T^{\lambda\mu\nu} + 64\xi A_\mu A^\mu + 8\zeta T^\mu A_\mu) - \right. \\
\left. -8(\zeta T^\mu + 16\xi A^\mu)\beta\partial_\mu\beta + 64\xi g^{\mu\nu}(\partial_\mu\beta)(\partial_\nu\beta) + \Lambda\beta^4\right)\ .
\end{gathered}$$

Here $\bar{R}^a{}_{b\mu\nu}$ is the Weyl–Cartan curvature tensor, $\bar{R}$ is the Weyl–Cartan curvature scalar, and the contractions of the Greek indexes are performed with the Riemann–Cartan metric tensor $g_{\mu\nu}$.

The given Lagrangian has some distinctive features. First, it reproduces the quadratic Lagrangian of the Poincaré-gauged theories of gravity [7], [12], [15]. Second, these Lagrangian, despite the gauge invariance, permits the presence of nonzero mass of the Weyl vector and therefore of the dilatation gauge field, in contrast to [24], [25]. This circumstance means that the gauge field, introduced by localization of the group of scale transformations, is not an electromagnetic field (as against initial idea of Weyl and from [42]), but a field of other nature, what was pointed to in [39]–[41]. A nonzero mass of the Weyl field can play a positive role in interpretations of the modern observational data on the basis of using post-Riemannian cosmological models [34], [35], and also for a possible explanation of a graceful exit from a stage of inflation. Besides, the last terms with a field $\beta(x)$ in this Lagrangian have the structure of the Higgs Lagrangian [42] and can play a determining role at spontaneous violation of scale invariance and formation of mass of particles [30].

Authors thank I. V. Tyutin for fruitful discussion of results of the work.

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